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Notes on discrete subgroups of $PU(1,2;\mathbf{C})$ with Heisenberg translations II

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In a previous paper [8] we have seen that under some conditions Parker's theorem yields the discreteness condition of Basmajian and Miner for groups with a Heisenberg translation. We show that we can obtain the same result as in [8] without the assumption on r.

1. First we recall some definitions and notation. Let $\mathbf{C}$ be the field of complex numbers. Let $V=V^{1,2}(\mathbf{C})$ denote the vector space $\mathbf{C}^3$, together with the unitary structure defined by the Hermitian form

$$\widetilde{\Phi}(z^*,w^*)=-(\overline{z_0^*}w_1^*+\overline{z_1^*}w_0^*)+\overline{z_2^*}w_2^*$$

for $z^*=(z_0^*,z_1^*,z_2^*), w^*=(w_0^*,w_1^*,w_2^*)$ in V. An automorphism g of V, that is a linear bijection such that $\widetilde{\Phi}(g(z^*),g(w^*))=\widetilde{\Phi}(z^*,w^*)$ for z^*,w^* in V, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,2;\mathbf{C})$. Let $V_0=\{w^*\in V|\quad \widetilde{\Phi}(w^*,w^*)=0\}$ and $V_-=\{w^*\in V|\quad \widetilde{\Phi}(w^*,w^*)<0\}$. It is clear that V_0 and V_- are invariant under $U(1,2;\mathbf{C})$. We denote $U(1,2;\mathbf{C})/(center)$ by $PU(1,2;\mathbf{C})$. Set $V^*=V_-\cup V_0-\{0\}$. Let $\pi: V^*\longrightarrow\pi(V^*)$ be the projection map defined by $\pi(w_0^*,w_1^*,w_2^*)=(w_1,w_2)$, where $w_1=w_1^*/w_0^*$ and $w_2=w_2^*/w_0^*$. We write ∞ for $\pi(0,1,0)$. We may identify $\pi(V_-)$ with the Siegel domain

$$H^2=\{w=(w_1,w_2)\in\mathbf{C}^2\quad|\quad Re(w_1)>\frac{1}{2}|w_2|^2\}.$$

We can regard an element of $PU(1,2;\mathbf{C})$ as a transformation acting on H^2 and its boundary ∂H^2 (see [6]). Denote $H^2\cup\partial H^2$ by $\overline{H^2}$. We define a new coordinate system in $\overline{H^2}-\{\infty\}$. Our convention slightly differs from Basmajian-Miner [1] and Parker [8]. The *H-coordinates* of a point $(w_1,w_2)\in\overline{H^2}-\{\infty\}$ are defined by $(k,t,w_2)_H\in(\mathbf{R}^+\cup\{0\})\times\mathbf{R}\times\mathbf{C}$ such that $k=Re(w_1)-\frac{1}{2}|w_2|^2$ and $t=Im(w_1)$. For simplicity, we write $(t_1,w')_H$ for $(0,t_1,w')_H$. The *Cygan metric* $\rho(p,q)$ for $p=(k_1,t_1,w')_H$ and $q=(k_2,t_2,W')_H$ is given by

$$\rho(p,q)=|\{\frac{1}{2}|W'-w'|^2+|k_2-k_1|\}+i\{t_1-t_2+Im(\overline{w'}W')\}|^{\frac{1}{2}}.$$

We note that the Cygan metric ρ is a generalization of the *Heisenberg metric* δ in ∂H^2 (see [7]).

Let $f = (a_{ij})_{1\le i,j\le 3}$ be an element of $PU(1,2;\mathbf{C})$ with $f(\infty) \neq \infty$. We define the *isometric sphere* I_f of f by

$$I_f = \{w = (w_1, w_2) \in \overline{H}^2 \mid |\tilde{\Phi}(W,Q)| = |\tilde{\Phi}(W, f^{-1}(Q))|\},$$

where $Q = (0,1,0)$, $W = (1, w_1, w_2)$ in V^* (see [4]). It follows that the isometric sphere I_f is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,

$$I_f = \left\{ z = (k,t,w')_H \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C} \mid \rho(z, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

2. We shall give a modified version of the stable basin theorem of Basmajian and Miner in [1]. Let

$$B_r = \{z \in \partial H^2 \mid \delta(z,0) < r\},$$

and let $\overline{B}_s^c = \partial H^2 - \overline{B}_s$. Given r and s with $r < s$, the pair of open sets $(B_r, \overline{B}_s^c)$ is said to be *stable* with respect to a set S of elements in $PU(1,2;\mathbf{C})$ if for any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}_s^c.$$

A loxodromic element f has a unique complex dilation factor $\lambda(f)$ such that $|\lambda(f)| > 1$. Let $S(r,\varepsilon)$ denote the family of loxodromic elements f with fixed points in B_r and $\overline{B}_{1/r}^c$, and satisfying $|\lambda(f) - 1| < \varepsilon$. For positive real numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$, we define $\varepsilon(r,r')$ by

$$\varepsilon(r,r') = \sup\{|\lambda(f) - 1|\}, \tag{2.1}$$

where $|\lambda(f) - 1|$ satisfies the inequality

$$|\lambda(f) - 1| < \sqrt{2 + \left(\frac{1-(3+|\lambda(f)-1|)r^2}{1-2r^2}\right)^2 \left(\frac{1-3r^2}{1-r^2}\right)^2 \left(\frac{r'}{r}\right)^2} - \sqrt{2}. \tag{2.2}$$

A triple of non-negative numbers (r, r', ε) is said to be a *basin point* provided that $r < 1/\sqrt{3}$, $r' < 1$ and $\varepsilon < \varepsilon(r,r')$. In particular, if $r' \le r$, we call (r,r',ε) a *stable basin point.* Call the set of all such points the *stable basin region.* For simplicity, we abbreviate (r,r,ε) to (r,ε)

Theorem 2.1 ([8; Theorem 3.1]). *Given positive real numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$, the pair of open sets $(B_{r'}, \overline{B}_{1/r'}^c)$ is stable with respect to the family $S(r, \varepsilon(r,r'))$, where $\varepsilon(r,r')$ is given by (2.1).*

3. We begin with introducing Parker's theorem on the discreteness of subgroups of $PU(1,2;\mathbf{C})$.

Theorem 3.1 ([9; Theorem 2.1]). *Let g be a Heisenberg translation with the form*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \overline{a} \\ a & 0 & 1 \end{pmatrix},$$

where $Re(s) = \frac{1}{2}|a|^2$. Let f be any element of $PU(1,2;\mathbf{C})$ with isometric sphere of radius R_f. If

$$R_f^2 > \delta(gf^{-1}(\infty), f^{-1}(\infty))\delta(gf(\infty), f(\infty)) + 2|a|^2,$$

then the group $< f, g >$ generated by f and g is not discrete.

Remark 3.2. Suppose that g is a vertical Heisenberg translation. As $a = 0$, this theorem is equivalent to results in [5] and [6].

In Theorem 4.5 of [8] we have shown that if $r < 0.484$, then Theorem 3.1 leads to the discreteness condition of Basmajian and Miner for groups with a Heisenberg translation. By using a more precise estimate on the Heisenberg distance between fixed points of f in terms of R_f and $\lambda(f)$, we have the following same result without the assumption on r.

Theorem 3. 3. *Fix a stable basin point (r, ε). Let g be the same element as in Theorem 3.1. Let f be a loxodromic element with fixed point 0 and q, and satisfying $|\lambda(f) - 1| < \varepsilon$. If $\delta(0,q) > \frac{\delta(0,g(0))}{r^2}(1 + r^2 + \sqrt{1 + r^2})$, then the group $< f, g >$ generated by f and g is not discrete.*

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